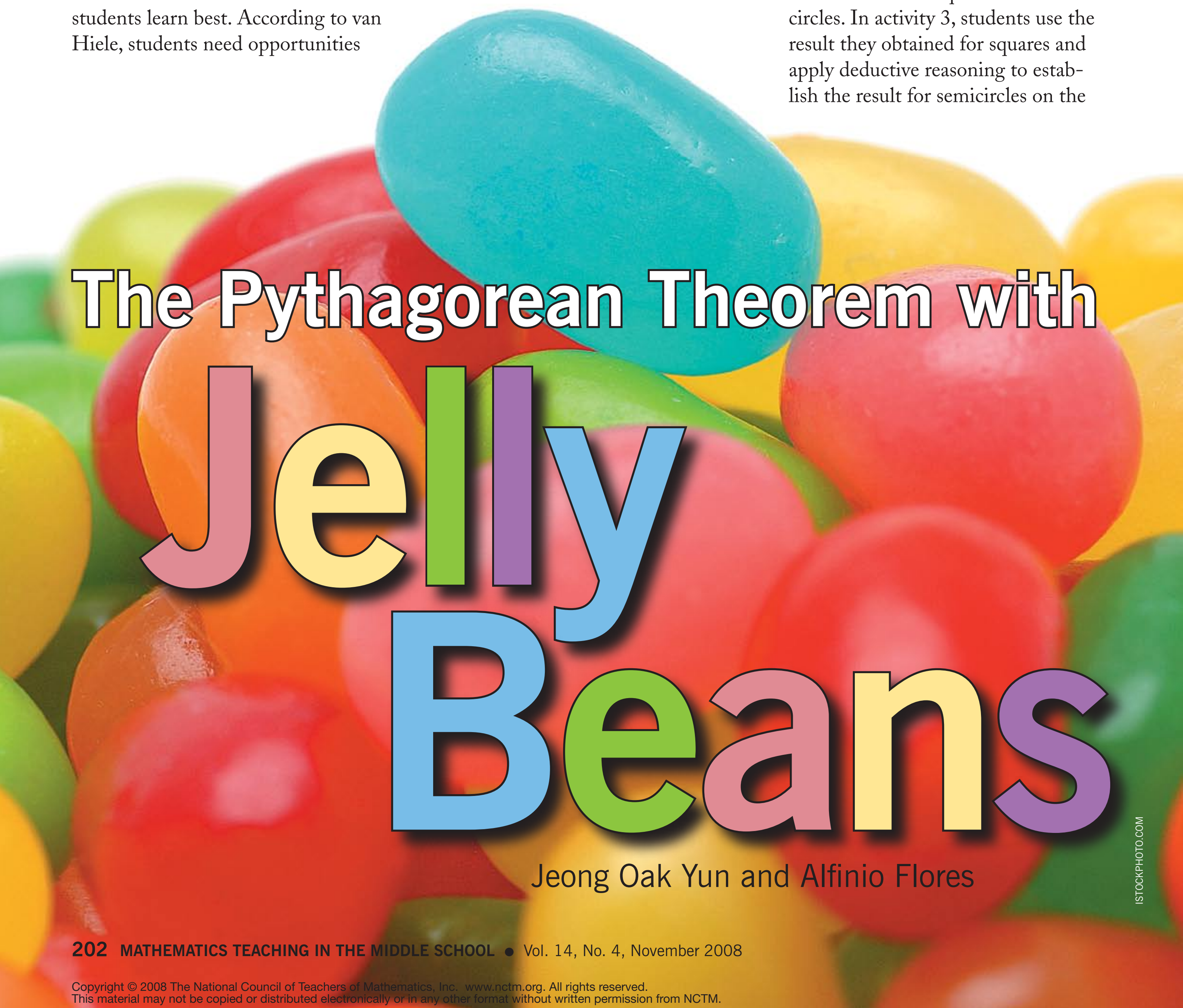
The Pythagorean Theorem with Jelly Beans

Jeong Oak Yun and Alfinio Flores



Principles and Standards for School Mathematics advocates an experimentation approach to middle-grades geometry study (NCTM 2000). Students are asked to explore and examine a variety of geometric shapes and discover their characteristics and properties using hands-on materials. They also create inductive arguments about the Pythagorean relationship. This empirical approach to the Pythagorean theorem, for example, will lay the foundation for analytical proofs.

Incorporating experimentation as part of middle school geometry is also consistent with research on how students learn best. According to van Hiele, students need opportunities to develop their geometric thinking through five levels: (1) visualization, (2) analysis, (3) informal deduction, (4) deduction, and (5) rigor (Fuys, Geddes, and Tischler 1988; van Hiele 1986). Levels 2 and 3, analysis and informal deduction, are especially relevant in the middle grades. At the analysis level, students can go beyond their perception of shapes and analyze the components, parts, and properties of shapes. Students use descriptions instead of definitions. They also discover and prove properties or rules by folding, measuring, or using a grid or a diagram. At the informal deduction level, students learn the role of definitions, analyze the relationships between figures, order figures hierarchically according to their characteristics, and deduce facts logically from previously accepted facts using informal arguments. The activities presented here will help students make the transition from level 2, analysis, to level 3, informal deduction.

In activities 1 and 2, middle-grades students explore the Pythagorean theorem by using jelly beans to estimate the areas of squares and semicircles. In activity 3, students use the result they obtained for squares and apply deductive reasoning to establish the result for semicircles on the

Fig. 1 The mat

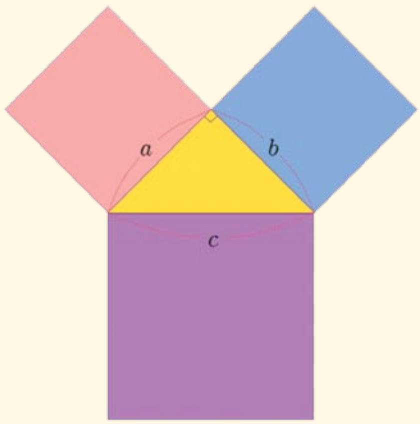
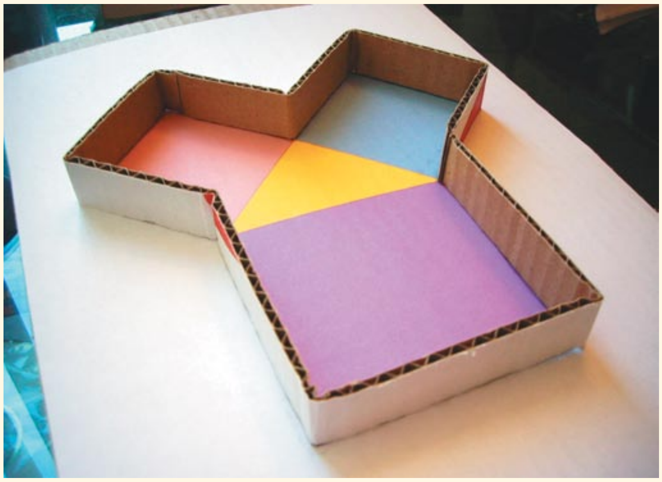
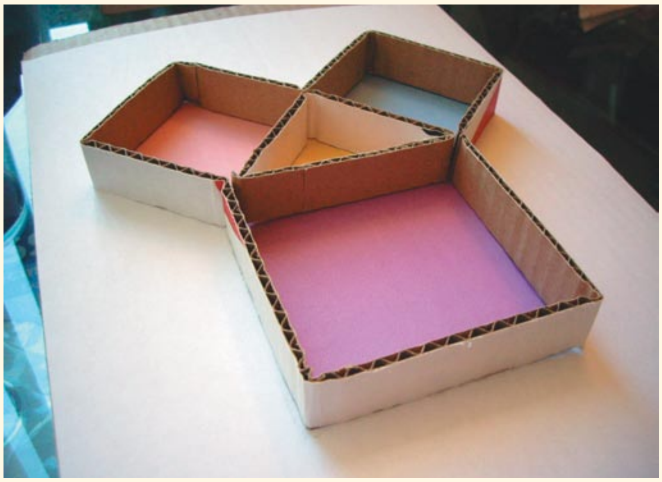


Fig. 2 The jelly bean structure



(a)
The mat's outside fence is glued in place.



(b)
The triangular fence is added but not glued.

(c)
A layer of jelly beans covers the legs.

sides of the right triangle. In activity 4, students use their discovery about the areas of semicircles and deductive reasoning to establish further relations among figure areas.

THE JELLY BEAN STRUCTURE

Activity 1 uses a cardboard mat that is constructed with sides to hold jelly beans. **Figure 1** shows a right triangle and its corresponding squares. **Figure 2** illustrates the mat with cardboard fencing to hold one layer of jelly beans.

To construct the mat, draw the shapes by hand or with computer software and paste the colored figures on cardboard that is about the size of a shoe box. To make the fences, cut strips of corrugated cardboard that are about 2 cm wide. (We recommend cutting perpendicularly to the ridges

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Fig. 3 A scalene right triangle

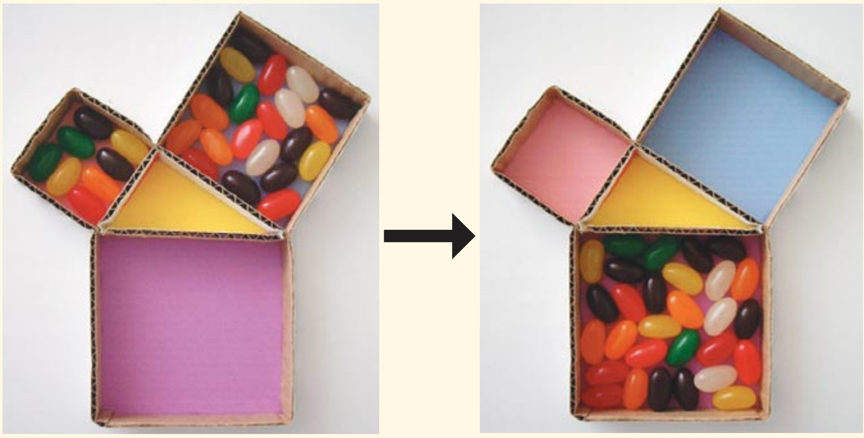


Fig. 4 Jelly beans on the move

(a)
Slide the jelly beans from the squares on the two legs to the square on the hypotenuse.

(b)
Jelly beans cover the square on the hypotenuse.

so that the strips can be bent easily.) Paste the strips on the mat along the outside of the figures with instant glue. Make the fence for the inner right triangle separately and *do not glue it down*, since it will be removed during the experiment. Estimating the area with jelly beans works well as long as students abstain from eating the candy because (1) jelly beans are easy to layer; (2) empty space, or gaps, around the jelly beans will be kept to a minimum; and (3) students like the bright colors. Although jelly beans are three-dimensional objects, we are using one layer, which is two dimensional. In this activity, students will not have to count the number of jelly beans to compare areas. The activities are designed to allow students to move jelly beans from one section of the mat to another to see whether or not all the jelly beans fit into a new shape.

There are several ways in which the activities could be conducted. The teacher can demonstrate the first activity and discuss the relationship between the areas of the squares. Students can also work with the mat in groups. Different groups could be given different mats based on different shapes, such as isosceles right triangles (see **fig. 2c**) and scalene right triangles (**fig. 3**), and then compare results. Students could move from one station to the next, where the layout of each mat is different, and conduct a similar experiment.

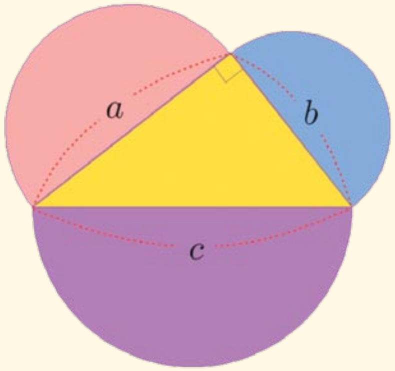
ACTIVITY 1: THE PYTHAGOREAN RELATIONSHIP

Students will use a mat that consists of a right triangle with squares on each of the three sides. With the cardboard fence glued around the three squares, students insert the triangular fence around the triangle (see **fig. 2b**). They should notice that a square is formed on each side of the right triangle. Students fill the two squares on the legs of the right triangle with jelly beans, ensuring that jelly beans completely fill the two squares without overlapping, forming just one layer (see **fig. 2c**). Next, students remove the cardboard triangle that is inside the frame and push all the jelly beans into the square on the hypotenuse (see **fig. 4a**). Finally, they re-insert the triangular fence and flatten the layer of jelly beans. Students verify whether the jelly beans completely cover the square on the hypotenuse (see **fig. 4b**). The teacher guides students to explicitly express in their own words what they found about the sum of the areas of the squares on the legs of the right triangle compared with the area of the square on the hypotenuse. Then students label the two legs of the right triangle as a and b, and the hypotenuse as c, and express the relationship using an algebraic equation: $a^2 + b^2 = c^2$.

ACTIVITY 2: EXTENDING THE PYTHAGOREAN THEOREM

Students can also use jelly beans to explore the relationships between areas when other similar shapes are constructed on the sides of a right triangle. Students use a mat with

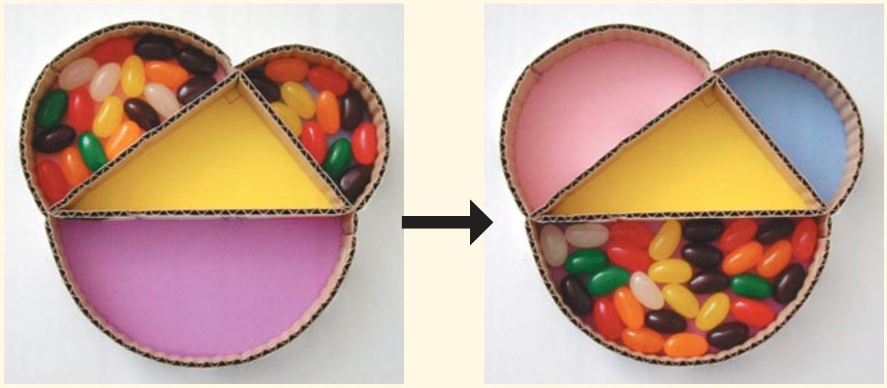
Fig. 5 Precise semicircles



(a)
Three semicircles surround a right triangle.

(b)
Three fences form three semicircles.

Fig. 6 Extending the Pythagorean theorem for semicircles



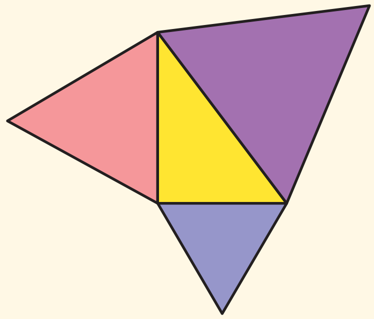
three semicircles adjacent to each side of a right triangle (see **fig. 5a**). The diameters of the semicircles are the sides of the right triangle. With the cardboard fence around the three semicircles glued to the mat, students insert the triangular fence around the right triangle. They need to notice that the two fences together form three semicircles on the sides of the right triangle (see **fig. 5b**). When making the structure for students, be sure that the arcs are precise semicircles. In the same way as in the first activity, students pour one layer of jelly beans in the two semicircles on the legs of the right triangle, completely covering the two semicircles (see **fig. 6**). Then they remove the cardboard triangle that is inside the frame and slide the jelly beans into the semicircle on the hypotenuse. Next, they re-insert the triangular fence, flatten the layer of jelly beans, and verify whether the jelly beans completely cover the semicircle on the hypotenuse. Students can describe in their own words the relationship they see between the sum of the areas of the semicircles on the legs of the right triangle and the area of the semicircle on the hypotenuse.

The teacher will also guide students to see that in each of the activities, the three shapes on the sides of the right triangle are similar to each other, that is, squares are similar to other squares, and semicircles are similar to other semicircles. These two activities correspond to van Hiele's level 2, because the verifications are done empirically.

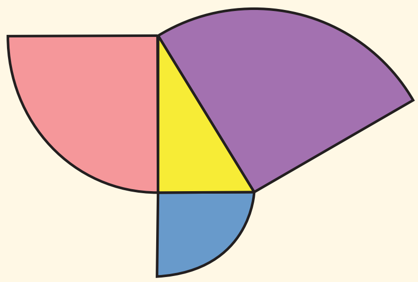
Other Extensions

Students can also experiment with different shapes constructed on the sides of the right triangle, as long as all three shapes are similar and their corresponding sides are placed on the sides of the right triangle (see **figs. 7a** and **7b**). Of course, these extensions to the Pythagorean theorem in terms of similar shapes on the sides of a right triangle are not new (Flores 1992; Heath 1956; Pólya 1948), but students are usually surprised that the Pythagorean relationship holds true for shapes other than squares.

Fig. 7 Other extensions of the Pythagorean theorem



(a)
An equilateral triangle is along each side.



(b)
A quarter circle is along each side.

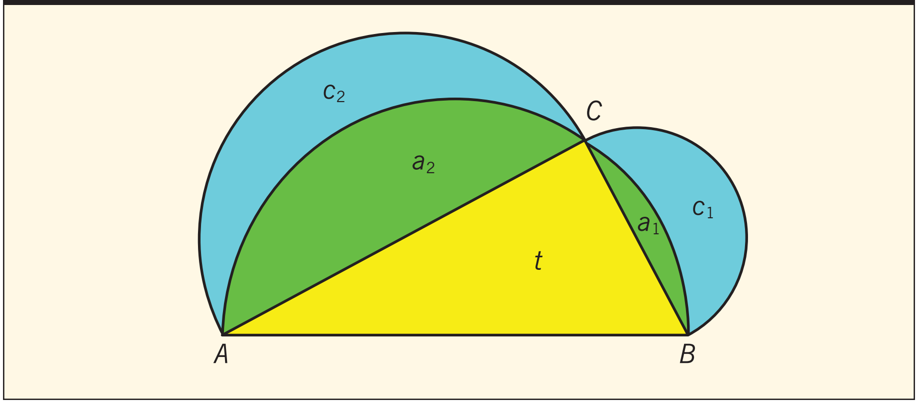
ACTIVITY 3: FORMULAS FOR THE AREAS OF THE SEMICIRCLES AND CONNECTIONS WITH ALGEBRA

Students can derive the extension of the Pythagorean theorem with semicircles by using some algebraic skills, such as factoring and simplifying algebraic expressions. They can do this activity working in pairs or small groups. Students can denote the two legs of the right triangle as a and b, and the hypotenuse as c (see **fig. 5a**) and express the relationship among the areas of the squares on the sides of the right triangle as $a^2 + b^2 = c^2$. They need also remember that the diameter of each semicircle is congruent to the corresponding side of the triangle and that the sum of the areas of the semicircles on the legs of the right triangle is equal to the area of the semicircle on the hypotenuse. The teacher can guide students to write algebraic expressions for the areas of the semicircles and for their relationships. Students can write an algebraic expression for each radius of the semicircles in terms of the length of each side of the triangle and write an expression for the area of each semicircle. They can use algebraic notation to state that the sum of the areas of the semicircles on legs a and b of the right triangle is equal to the area of the semicircle on the hypotenuse, c:

$$\frac{1}{2}\times\left(\frac{a}{2}\right)^2\pi+\frac{1}{2}\times\left(\frac{b}{2}\right)^2\pi=\frac{1}{2}\times\left(\frac{c}{2}\right)^2\pi$$

Students then factor and simplify both sides of the equation to show that it is equivalent to the equation $a^2 + b^2 = c^2$. To verify, they can reverse all the steps; starting with $a^2 + b^2 = c^2$, they can show how to obtain from this equation the relation among semicircles. This activity corresponds to van Hiele's level 3 because students make deductive arguments based on previously accepted facts.

Fig. 8 Crescents formed by semicircles



ACTIVITY 4: AREA OF THE CRESCENTS OF HIPPOCRATES

Hippocrates of Chios, a Greek mathematician from the fifth century BC, discovered that two figures bounded by arcs of circles have the same area as a figure bounded by straight lines. In **figure 8**, semicircles are constructed on the sides of right triangle *ABC*. However, the semicircle on the hypotenuse, rather than being on the outside of the triangle as in activity 2, is drawn on the inside. Students who have difficulty seeing this semicircle could look at three semicircles on the outside of a right triangle, as in **figure 5a**, and then think of reflecting the largest semicircle across the hypotenuse. The semicircle on the hypotenuse overlaps with the triangle (in yellow) and also overlaps partially with the semicircles on the legs (shown in green). The part of the semicircle on each leg that is completely outside the semicircle on the hypotenuse makes a crescent (shown in blue). Students can express the area of the semicircle on the hypotenuse as $t + a_1 + a_2$ (see **fig. 8**). The areas of the semicircles on the legs are $c_1 + a_1$ and $c_2 + a_2$. Using their discovery that the area of the semicircle on the hypotenuse is equal to the sum of the areas of the semicircles on the legs of the right triangle and the expression of areas stated above, students can find a relation between the sum of the areas of the two crescents and the area of the right triangle *ABC*. This activity also corresponds to level 3's informal deduction.

CONCLUDING REMARKS

Many students find that they dislike mathematics when they are asked to simply memorize what they cannot verify and understand on their own. When the Pythagorean theorem is first introduced using a deductive approach, students often have difficulty following the steps of the proofs. Even when they manage to follow each step to reach the conclusion, many students do not understand what it means. An approach that proves the Pythagorean theorem using only axioms or previously demonstrated theorems corresponds to *axiomatic deduction*, which is consistent with van Hiele's level 4. Students experience difficulties in a deductive presentation because level 4 work does not correspond to levels 2 and 3, where most students in the middle grades operate.

In the case of the Pythagorean theorem, students are sometimes confused about what areas they need to add to find a specific area measurement and what segments are being compared. There are computer programs that allow students to visualize the relationship. However, such programs usually ask students to simply believe the theorem, because all they can do is watch untouchable moving objects. The jelly beans activities are helpful for students because they can experiment directly with concrete objects to understand the Pythagorean theorem conceptually, rather than just memorizing or applying a formula. Students are less prone to forget what areas added together are equal to what other area. When the teacher prepares more activities using not only squares on the sides but also semicircles, students remember the mathematics better. They are surprised by the results and work to establish connections between the different cases by using algebraic notation.

These activities using jelly beans are beneficial not only for students. Prospective and in-service teachers also enjoy doing the activity. By removing the fence and sliding the jelly beans from the smaller shapes on the legs of the triangle to the bigger shape on the hypotenuse, they are verifying that indeed the shape gets covered (Yun 2007) and that there is a relationship among the sides and the hypotenuse of a right triangle.

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Editor's note: For other interactive approaches to the Pythagorean theorem, log on to *ON-Math*, NCTM's online journal, and read "The Pythagorean Theorem: Going beyond $a^2 + b^2 = c^2$" by Piatek-Jimenez.