Aperiodic Tiling

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From the pictures of Escher to the geometric patterns of Islamic art, repeating patterns and tiling have been used to generate many beautiful images. They have also been considered as part of mathematics for a long time, back at least to the ancient Greeks. More recently the study of tilings has been instrumental in understanding symmetry, which has led to group theory and key results in understanding the structure of crystals. All these results, however, have only considered periodic patterns. Since the 1960's a new area has emerged; the study of tilings and patterns that are ordered but not periodic. This is the area of aperiodic tilings.

A key question in the study of tilings is to decide whether a set of tiles can tile the plane. The simplest way to do this is to exploit periodicity. A *periodic* tiling repeats in such a way that it can be picked up, moved some distance and put down, so it fits exactly on top of itself. It is hard to draw tilings covering an infinite plane. So, we work with small regions of tiling, called *patches*. Patches of periodic tilings are everywhere, from the square tiles in bathrooms and on chess boards, to bricks and honeycombs. Another example is shown in Figure 1. To show that a set of tiles can tile periodically one finds a patch of tiles that fits together with itself, as shown in Figure 2. For most of the history of tiling, the tilings considered were periodic. In fact, most people believed that every set of tiles that could tile the plane could do so in a periodic way.

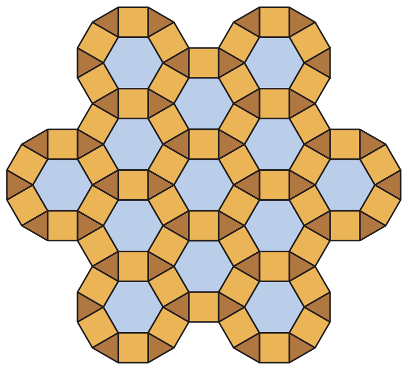

Figure 1: A periodic tiling of the plane with regular hexagons, squares and equilateral triangles.

In 1966, following a question of Wang **[Wan61]**, Robert Berger **[Ber66]** showed that this is not the case. Berger produced an *aperiodic set of tiles*, that is a set of tiles that can tile the plane, but not periodically, but it had 20426 different tiles! This number was gradually reduced until, in the 1970's, Roger Penrose discovered the famous Penrose tiling (announced by Martin Gardner in **[Gar77]**), with only two different tiles. This beautiful tiling forms the background for this poster. The question of whether a single aperiodic tile exists is till open.

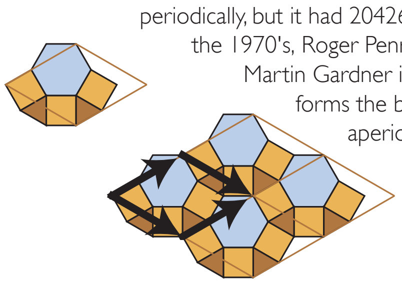

Figure 2: Using periodicity to show that a tiling tiles the plane.

How does one show that a set of tiles can tile the plane without using periodicity? The answer is a system called a substitution rule, described here for the Penrose Rhombs. Substitution rules are important in their own right, as examples of scaling symmetry. The work of H and L is starting to give a detailed theory of these fascinating tilings.

Penrose Rhombs

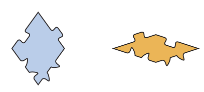

Figure 5: The Penrose Rhombs.

The two Penrose Rhomb tiles are shown in Figure 5. They tile the plane, but not periodically. How then does one show that they can tile the plane? The answer is a trick called a *substitution rule*, illustrated below. Substitution rules are in fact tightly linked to aperiodic tilings, Goodman-Strauss **[GS98]** showed that any substitution rule satisfying some basic conditions could be used to find an set of aperiodic tiles.

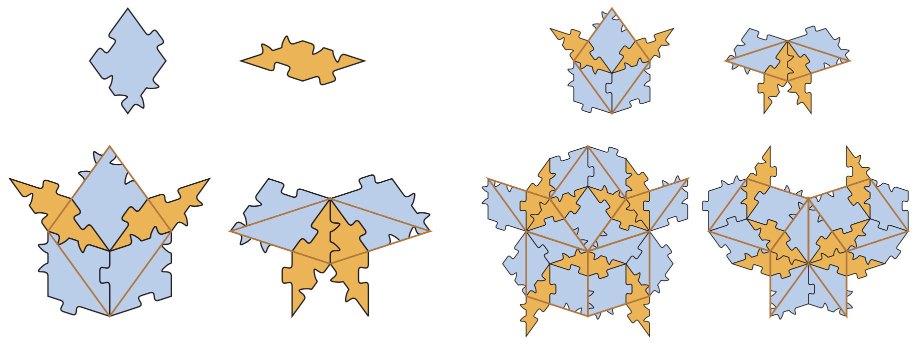

Figure 6: Take two patches of the Penrose tiles (a). These patches can be considered as larger tile (or super tiles) as they fit together in the same way as the original two tiles (b). This gives some overlaps but the overlapping tiles overlap entirely. We may therefore build up the same patches with these larger tiles (c). The larger tiles can then be replaced by their respective patches (d).

Such a system is called a substitution rule. In general it consists of two stages. Firstly the tiles are enlarged (getting the larger tiles in (b)) and then the larger tiles are replaced by patches of the original tiles. Repeating this process gives larger and larger patches. This shows that the tiles will tile the whole plane.

Quasicrystals

In crystallography, something is considered to be ordered and a crystal if its diffraction pattern has sharp peaks (called Bragg peaks). It had always been assumed that such structures had to be periodic. In three dimensions, the only possible rotational symmetries for a periodic structure are two, three, four and six. Thus it was a shock when, in 1984, Shechtmann et al. **[SBGC84]**, found metallic crystals whose diffraction pattern had five fold rotational symmetry.

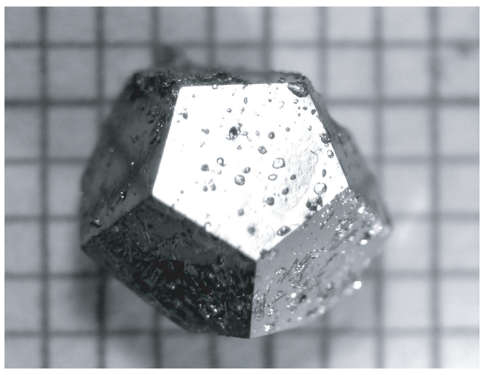

Figure 4: An Al-Pd-Re quasicrystal. Image from **[FXetal02]** courtesy of I.R. Fisher, Stanford.

This opened the way to a new class of crystals, named *quasicrystals*. These materials have an ordered structure and their diffraction patterns have sharp peaks, but are not periodic.

Interest in aperiodic tilings really exploded with this discovery. The Penrose tiling (and its three dimensional analogue) were found to be one of the best models for the structure of these quasicrystals.

In fact the same tilings and patterns turn up in many areas of physics where aperiodic order is being discovered, another example is in Faraday wave patterns **[EF94]**.

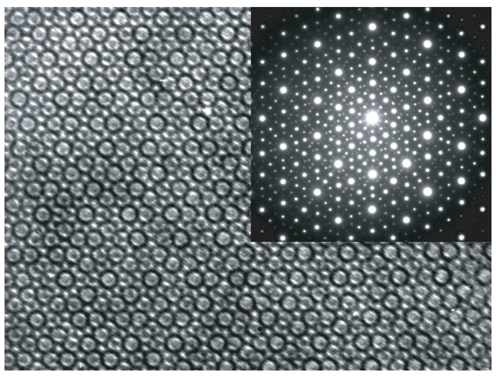

Figure 5: TEM micrograph and diffraction pattern (inset) of an Al-Pd-Re quasicrsytal, both along an axis of five-fold symmetry Image from **[FXetal02]** courtesy of I.R. Fisher, Stanford.

Canonical Substitution Tilings

In 1981, De Bruijn **[DeB81]** showed that the Penrose tiling had a second construction, as the projection of a slice of a five dimensional lattice. This led to the development of a general construction, known as the *canonical projection method,* that has been extensively studied in the quest to understand the mathematics behind quasicrystals and aperiodic order **[BM00]**.

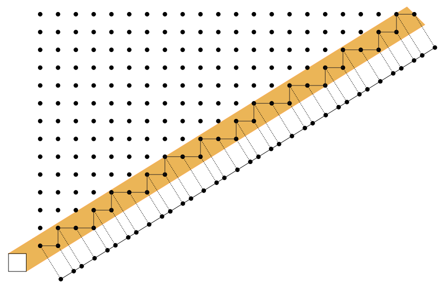

Figure 7: The Fibonacci tiling of the line, described in the text. The lattice is the set of dots. The sloping line is **S**, the square U and the orange strip **U+S**. The tiling is given on the line with two tiles.

For the canonical projection method one starts with a lattice **L** in an n-dimensional space, and an m-dimensional subspace **S**. One then considers the region given by all possible translations of the unit cell of the lattice, **U** along **S**, (**U+S**). The intersection of this strip with the lattice gives a set of points. The projection of this set of points, connected by the lattice generators, gives a tiling of the subspace. If the subspace only intersects the lattice at **0**, it is called *totally irrational* and the tiling will be non-periodic. This is perhaps better explained with an example shown in Figure 7.

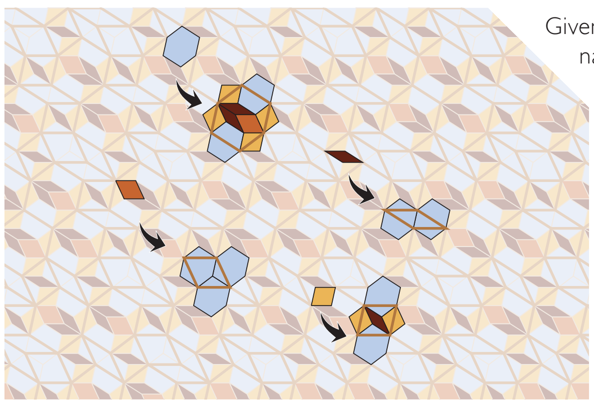

Figure 8: A substitution tiling found using the projection method and the results of H and L.

Given this general construction a natural question is to ask if there are other examples, like the Penrose which can also have a substitution rule. Recently the work of H and L has given a complete characterisation of all such tilings **[Har03]**, giving many new examples, such as Figure 8. This result is also opening up the study of substitution tilings, being the first step in the characterisation of all such tilings.

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