A Connection between the Numbers Phi and Pi

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Draft, 2008 January 21

Abstract

In this paper, we explore an interesting connection between the two fundamental mathematical constants, pi π and phi ϕ, by means of the regular polygons with an odd number of sides and the Golden Numbers, which we defined in our earlier paper [Anderson and Novak (2008)]. We note that phi is the smallest of the Golden Numbers and show that pi is related to the largest of them. Because the Golden Numbers may be interpreted geometrically as the ratios of the lengths of the diagonals to the lengths of the sides of odd-sided regular polygons, this connection between phi and pi has a geometric interpretation. As phi is the ratio of the diagonal to the side (one-fifth the periphery) of a regular pentagon, so pi is the limit of the ratio of the periphery to the longest diagonal of polygons as the number of their sides increases without limit. We define several functions of the number of sides of regular polygons that describe the properties of polygons and study them as this number increases from 5 to ∞, as a polygon is transformed from a pentagon to a circle.

1 Introduction

The two real numbers π and ϕ are of fundamental importance in mathematics. The purpose of this paper is to show how they are related to each other numerically and geometrically. We shall show that the two numbers are connected numerically by means of the Golden Numbers, a set of real numbers that we defined in our paper [Anderson and Novak (2008)] on generalized Fibonacci sequences and regular polygons. The number ϕ, generally known as the Golden Ratio, is simply the smallest of the Golden Numbers. The number π, the ratio of the circumference to the diameter of a circle, is related to the largest of the Golden Numbers.

The two numbers are connected to one another geometrically by means of the sequence of regular polygons with an odd number $2n + 1$ of sides, where n ranges from 2 to ∞. The number ϕ is defined in the case $n = 2$ of the regular pentagon, which has five $2n + 1 = 5$ sides. It is the ratio of the length of a diagonal to the length of a side or, thus, $2n+1$ times the diagonal divided by the periphery. The number π is defined at the other end of the sequence, as the number of sides of a polygon increases without limit and the polygon approximates a circle. It is the limit of the ratio of the periphery (approximating the circumference) to the longest diagonal (approximating the diameter) or, thus, the limit of $2n+1$ times the side divided by the longest diagonal.

In the next Section, we review our work on the Golden Numbers and show that the Golden Ratio ϕ is the smallest of them. In the last Section, we define several quantities that pertain to regular polygons of $2n+1$ sides and study how they vary as functions of the order number n. We find that, over the range from $n = 2$ to $n = \infty$, these functions all increase slowly, by less than 7%. The function that we call the pseudopi function varies the least, by less than 2%, from $\frac{5}{\phi}$ at $n = 2$ to π at $n = \infty$, demonstrating an intriguing relationship between π and ϕ, two numbers of fundamental mathematical importance.

2 The Golden Numbers

The Fibonacci sequence is related to the geometry of the regular pentagon by means of the Golden Ratio ϕ, which is the ratio of the length of a diagonal to the length of a side and has the approximate value 1.618033989. Following the work of George Raney [Raney (1966)], we showed in an earlier paper [Anderson and Novak (2008)] that the generalized Fibonacci sequences of order n are related to the regular polygons having $2n+1$ sides by a set of $n-1$ numbers, $r_n(i)$, where $i = \{2, \ldots, n\}$, that we call Golden Numbers. We derived a simple trigonometric formula to calculate the Golden Numbers.

In order to state this formula, let us summarize the relevant features of the geometry of regular odd-sided polygons. Such a polygon has $2n+1$ equal sides of length $d_n(1)$. It also has $2n+1$ diagonals of $n-1$ different lengths, $d_n(i)$, where i is the index of the diagonals, ranging from the shortest $d_n(2)$ to the longest $d_n(n)$. All the angles between the diagonals and the sides of such a polygon are integral multiples of a minimum angle α_n

$$\alpha_n = \frac{\pi}{2n+1} \tag{2.1}$$

where $\pi \approx 3.141592654$ is the equivalent in radian measure of $180°$.

We can now state the formula for the Golden Numbers. The Golden Number $r_n(i)$ is the ratio of the length of a diagonal of index i, $d_n(i)$, to the length of a side $d_n(1)$ of a regular polygon having $2n+1$ sides.

$$r_n(i) = \frac{d_n(i)}{d_n(1)} = \frac{\sin i\alpha_n}{\sin \alpha_n} = \frac{\sin(\frac{i\pi}{2n+1})}{\sin(\frac{\pi}{2n+1})} \tag{2.2}$$

The smallest Golden Number $r_2(2)$ is the Golden Ratio ϕ.

$$\begin{aligned} r_2(2) &= \frac{d_2(2)}{d_2(1)} = \frac{\sin 2\alpha_2}{\sin \alpha_2} = \frac{\sin \frac{2\pi}{5}}{\sin \frac{\pi}{5}} \\ &= 2\cos\frac{\pi}{5} = 2\cos 36^\circ = \phi \approx 1.618033989 \end{aligned} \tag{2.3}$$

Our earlier paper contains a Table of the first few Golden Numbers, calculated from equation (2.2).

While the number ϕ is the smallest Golden Number, the number π is related to the largest Golden Numbers. As the number of sides $2n+1$ of the polygon increases without limit, the polygon approaches a circle. The periphery of the polygon c_n, which is $2n+1$ times the side length $d_n(1)$, approaches the circumference of the circle.

$$c_n = (2n+1) \cdot d_n(1) \tag{2.4}$$

The longest diagonal of the polygon $d_n(n)$, approaches the diameter of the circle. Thus, the number π, which is defined as the ratio of the circumference to the diameter of the circle, may be found in the limit as $n \to \infty$ of the ratio of the periphery to the longest diagonal of the polygon or, thus, as $2n+1$ times the reciprocal of the ratio of the longest diagonal $d_n(n)$ to the side $d_n(1)$, which is the largest Golden Number $r_n(n)$ for that odd-sided polygon of order n.

$$\pi = \lim_{n\to\infty} \frac{c_n}{d_n(n)} = \lim_{n\to\infty} \frac{(2n+1)\cdot d_n(1)}{d_n(n)} = \lim_{n\to\infty} \frac{2n+1}{r_n(n)} \approx 3.141592654 \tag{2.5}$$

The two fundamentally important real numbers ϕ and π are connected numerically by the Golden Numbers $r_n(i)$. The number ϕ is simply the smallest of them, $r_2(2)$, while π is related to the largest of them, as the limit as $n \to \infty$ of $2n+1$ times the reciprocal of the largest Golden Number for that value of n, $r_n(n)$.

Another way to represent this numerical connection is to consider the sequence of values of what we call the inverse of the pseudopi function of n, π_n^{-1}, defined as the ratio of the largest Golden Number of order n, $r_n(n)$, to the number of sides, $2n+1$.

$$\pi_n^{-1} = \frac{r_n(n)}{2n+1} \tag{2.6}$$

At the low end of its domain, for $n=2$, this function has the value

$$\pi_2^{-1} = \frac{\phi}{5} \approx 0.323606797 \tag{2.7}$$

At the the high limit of its domain, as $n \to \infty$, it has the value

$$\pi_\infty^{-1} = \frac{1}{\pi} \approx 0.318309886 \tag{2.8}$$

a decrease in value of about 1.6%. The number ϕ (divided by 5) is connected to (the reciprocal of) the number π by a chain, an infinite sequence of numbers that vary by less than 2%.

3 Functions of Polygons

In this section, we further explore the relationship between the two numbers π and ϕ by defining several functions of the order n that represent the properties of polygons and studying how they vary as the order n increases from 2, describing the pentagon, to ∞, describing the circle. As we vary the order n, there are at least two options. One, we keep the polygon's side length fixed; then, as n increases, the size of the polygon, as measured by the diameter of the circumscribed circle, increases without limit. Two, we keep the size of the polygon fixed, that is, we fix the diameter of the circumscribed circle; then, as n increases, the side length decreases toward 0. Here, we select the second option, because it is easier to compare polygons of the same size. We assume that the diameter d of the circumscribed circle is fixed for all n and that, therefore, the side length $d_n(1)$ varies as a function of n, and decreases to 0, as n increases without limit.

$$d_n(1) = d \cdot \sin \alpha_n = d \cdot \sin \frac{\pi}{2n+1} \tag{3.1}$$

Before proceeding further, let us first note a few useful relationships between the Golden Ratio ϕ and trigonometric functions of α_2, the minimal angle in the case $n = 2$ of the regular pentagon.

$$\phi = \frac{d_2(2)}{d_2(1)} = \frac{\sin 2\alpha_2}{\sin \alpha_2} = 2\cos \alpha_2 \tag{3.2}$$

$$\sin \alpha_2 = \sin \frac{\pi}{5} = \sin 36^\circ = \sqrt{1 - \frac{\phi^2}{4}} \approx 0.587785252 \tag{3.3}$$

$$\sin 2\alpha_2 = \sin \frac{2\pi}{5} = \sin 72^\circ = \phi \sqrt{1 - \frac{\phi^2}{4}} \approx 0.951056516 \tag{3.4}$$

We now define several variables that pertain to regular odd-sided polygons as functions of the order n and calculate their numerical values for $n = 2$ and for $n = \infty$. First, the circumference c_n function of n is the periphery, $2n + 1$ times the side length $d_n(1)$.

$$c_n = (2n + 1) \cdot d_n(1) = (2n + 1) \cdot \sin \alpha_n \cdot d \tag{3.5}$$

$$c_2 = 5 \sin \alpha_2 \cdot d = 5 \sqrt{1 - \frac{\phi^2}{4}} \cdot d \approx 2.938926262 \cdot d \tag{3.6}$$

$$c_\infty = \pi d \approx 3.141592654 \cdot d \tag{3.7}$$

The greatest diagonal length we call the pseudodiameter d_n function of n.

$$d_n = d_n(n) = d_n(1) \frac{\sin n\alpha_n}{\sin \alpha_n} = \sin n\alpha_n \cdot d \tag{3.8}$$

$$d_2 = d_2(2) = \sin 2\alpha_2 \cdot d = \phi \sqrt{1 - \frac{\phi^2}{4}} \cdot d \approx 0.951056516 \cdot d \tag{3.9}$$

$$d_\infty = d \tag{3.10}$$

The ratio of the circumference to the pseudodiameter we call the pseudopi π_n function of n.

$$\pi_n = \frac{c_n}{d_n} = \frac{(2n+1)\sin\alpha_n}{\sin n\alpha_n} \tag{3.11}$$

$$\pi_2 = \frac{5\,\sin\alpha_2}{\sin 2\alpha_2} = \frac{5}{\phi} \approx 3.090169944 \tag{3.12}$$

$$\pi_\infty = \pi \approx 3.141592654 \tag{3.13}$$

In defining a function of n that equals ϕ when $n = 2$, there are at least two options, the ratio of $d_n(n)$, the pseudodiameter, to either $d_n(1)$, the side length of the $2n+1$ sided polygon, or $d_2(1)$, the side length of the 5 sided polygon, the pentagon. We reject the former option, because, as n increases, it it unbounded. We select the latter option as our pseudophi ϕ_n function of n. Note that the pseudophi function is essentially the same as the pseudodiameter function, except for a normalizing factor.

$$\phi_n = \frac{d_n(n)}{d_2(1)} = \frac{\sin n\alpha_n}{\sin\alpha_2} = \frac{d_n}{d\sin\alpha_2} \tag{3.14}$$

$$\phi_2 = \phi \approx 1.618033989 \tag{3.15}$$

$$\phi_\infty = \frac{1}{\sin\alpha_2} = \frac{1}{\sqrt{1-\frac{\phi^2}{4}}} \approx 1.701301617 \tag{3.16}$$

Finally, let us examine the product of the pseudopi π_n and the pseudophi ϕ_n functions of n. Note that this product is essentially the same as the circumference function, except for a normalizing factor.

$$\pi_n \cdot \phi_n = \frac{(2n+1)\cdot\sin\alpha_n}{\sin\alpha_2} = \frac{c_n}{d\sin\alpha_2} \tag{3.17}$$

$$\pi_2 \cdot \phi_2 = \pi_2 \cdot \phi = 5 \tag{3.18}$$

$$\pi_\infty \cdot \phi_\infty = \pi \cdot \phi_\infty = \frac{\pi}{\sqrt{1-\frac{\phi^2}{4}}} \approx 5.34479666 \tag{3.19}$$

Compare these values with the product of the numbers pi π and phi ϕ.

$$\pi \cdot \phi = \pi_\infty \cdot \phi_2 \approx 5.083203692 \tag{3.20}$$

You will have noticed that all of these functions of n vary slowly. The value of each at $n = \infty$ is greater than that at $n = 2$ by less than 7%. The least rapidly varying is the pseudopi function, the value of which at $n = \infty$, π, is less than 2% greater than its value at $n = 2$, $\frac{5}{\phi}$.

4 Conclusion

In our earlier paper [Anderson and Novak (2008)] on generalized Fibonacci sequences and the geometry of polygons with an odd number $2n+1$ of sides, we defined the Golden Numbers $r_n(i)$. In this paper, we have shown that these Golden Numbers provide a means to connect the two fundamental real numbers, pi π and phi ϕ. Specifically, we show that what we call the inverse of the pseudopi function of the order n, defined as the ratio of $r_n(n)$ to $2n+1$, is a chain that links these two special numbers, as it decreases by less than 2% from its value at $n=2$, $\frac{\phi}{5}$, to its value as $n \to \infty$, $\frac{1}{\pi}$. We also studied several increasing functions of the order n that represent geometrical properties of odd-sided polygons. We find it interesting that they all vary so little over the domain from $n=2$ to $n=\infty$, and that the pseudopi function varies the least of all. We find it intriguing that the pseudopi and the pseudophi functions should be so closely related. This study suggests an interesting relationship between the two fundamental real numbers, π and ϕ, but we believe that it barely begins to reveal the depth of the relationship between these two absolutely fundamental real numbers.

References

[Anderson and Novak (2008)] Stuart Anderson and Dani Novak, "Generalized Fibonacci Sequences and Regular Polygons," TBD, **TBD**, TBD (2008).

[Raney (1966)] George N. Raney, "Generalization of the Fibonacci Sequence to n Dimensions," Can. J. Math., **18**, 332-349 (1966).